3.3 On equations Reducible into Clairaut's Equations

N.B.: There are certain equations which are not in Clairaut's form but by a suitable substitution they are reduced to Clairaut's equation.

Ex ① Solve: $y = 2px + y^2p^3$

Soln:- Given

$$y = 2px + y^2p^3 \quad \text{--- ①}$$

Put $y^2 = Y$ and $x = X$ so that

$$2y\frac{dy}{dx} = \frac{dY}{dX} \text{ and } dx = dX$$

$$\Rightarrow 2yp = \frac{dY}{dX} \Rightarrow 2yp = P \quad \left\{P = \frac{dY}{dX}\right\}$$

$$\Rightarrow p = \frac{P}{2y}$$

Eq ① becomes, $y = 2x \cdot \frac{P}{2y} + y^2 \cdot \left(\frac{P}{2y}\right)^3$

$$\Rightarrow y = \frac{P \cdot x}{y} + \frac{y^2 P^3}{8y^3}$$

$$\Rightarrow \cancel{y = \frac{1}{y}}$$

$$y = \frac{1}{y}\left(Px + \frac{P^3}{8}\right)$$

$$\Rightarrow y^2 = Px + \frac{P^3}{8}$$

$$\Rightarrow Y = PX + \frac{P^3}{8}$$

which is in Clairaut's form.

The general solution is

$$Y = cX + \frac{c^3}{8} \quad \text{i.e. } y^2 = cx + \frac{c^3}{8} \quad \text{--- ②}$$

To obtain singular solution,

Diff ② w.r.to c

$$0 = x + \frac{3c^2}{8} \Rightarrow c^2 = -\frac{8x}{3} \quad \text{--- ③}$$

Putting these value in ②

$$y^2 = cx + \frac{c^2 \cdot c}{8}$$

$$\Rightarrow y^2 = cx + \frac{c}{8}\left(-\frac{8x}{3}\right)$$

$$\Rightarrow y^2 = \frac{3cx - cx}{3} = \frac{2cx}{3}$$

$$\Rightarrow y^4 = \frac{4}{9}c^2x^2 \quad \{\text{squaring}\}$$

$$\Rightarrow y^4 = \frac{4}{9}x^2 \times \left(-\frac{8x}{3}\right) \quad \{\text{from ③}\}$$

$$\Rightarrow y^4 = \frac{-32x^3}{27}$$

$$\Rightarrow 27y^4 + 32x^3 = 0$$

This is the singular solution.

Ex (2) Solve : $x^2\left(y - x\frac{dy}{dx}\right) = y\left(\frac{dy}{dx}\right)^2$

Soln:- Given, $x^2(y - xp) = yp^2$ —— (i)

~~$\Rightarrow x^2 y - x^3 p = yp^2$~~

~~$\Rightarrow y - x^2 p + y$~~

Put $x^2 = X$ and $y^2 = Y$ so that

~~$2x\,dx = dX$ and $2y\,dy = dY$~~

$\therefore \frac{dY}{dX} = \frac{\cancel{2}y\,dy}{\cancel{2}x\,dx} \Rightarrow \frac{y}{x}p = P \Rightarrow p = \frac{Px}{y}$ $\left\{P = \frac{dY}{dX}\right\}$

Putting the value of p in (i)

$x^2\left(y - x.\frac{Px}{y}\right) = y.\left(\frac{Px}{y}\right)^2$

$\Rightarrow x^2\left(\frac{y^2 - Px^2}{\cancel{y}}\right) = \cancel{y} \times \frac{P^2 x^2}{\cancel{y^2}}$

$\Rightarrow \cancel{x^2}(y^2 - x^2 P) = \cancel{x^2}P^2$

$\Rightarrow y^2 = x^2 P + P^2$

$\Rightarrow Y = PX + P^2$ —— (ii) {which is in Clairaut's form}

$\therefore$ The general solution is

$Y = cX + c^2$ —— (iii) $\Rightarrow y^2 = cx^2 + c^2$ —— (iv)

In order to find singular solution, Roots of c be equal

$\therefore b^2 - 4ac = 0$ $\{c^2 + cx^2 - y^2 = 0\}$

$\Rightarrow (x^2)^2 - 4 \times 1 \times (-y^2) = 0$

$\Rightarrow x^4 + 4y^2 = 0.$

which is called singular solution

Ex (3) Solve : $(px - y)(x - py) = 2p.$

Soln: Do yourself.

Put $x^2 = X$ and $y^2 = Y$ ..

~~Ans: (1) The general solution~~ : $y = cx - \frac{2c}{1-c}$

~~The singular solution~~

Semester-03

#4 Trajecteries of a family of Curves

#4.1 Definition

Trajectory :- A Curve which cuts every member of a given family of curves according to a given rule is called a trajectory of the given family.

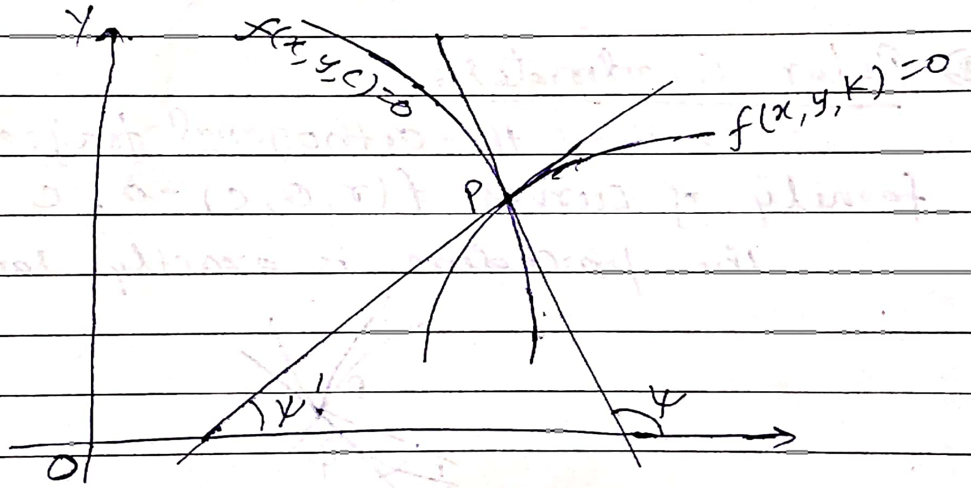
Every member of a given family of curves at a constant angle.

Orthogonal Trajectory :- If a constant angle is a right angle, then the trajectory is said to be orthogonal.

#4.2 Orthogonal Trajectories :-

(1) Cartesian Co-ordinates :-

To Find the orthogonal trajecteries of the family of curves $f(x,y,c)$; c being a parameter.



The equation of the curve, $f(x,y,c)=0$ —(1)

Diff (1) w.r. to x and eliminating c between (1) and the derived result.

Let the diff. eqⁿ. $\phi(x, y, \frac{dy}{dx}) = 0$ —(II) {using (1)}

Let $F(x,y,k)=0$ be the equation of the curve which cuts the given family $f(x,y,c)=0$ at a constant angle α.

Let P be the point of their intersection.

Let $m = \tan\psi = \frac{dy}{dx}$ and m' denotes the slope of tangents to the curves $f(x,y,c)=0$ and $F(x,y,k)=0$ resp.

Then, $\tan\alpha = \frac{m-m'}{1+mm'}$ —— (III)

If $\alpha = 90^\circ$, if the trajectories are orthogonal

then, $mm' = -1$

$$\Rightarrow m' = -\frac{1}{m} = \frac{-1}{\frac{dy}{dx}} = \frac{-dx}{dy}.$$

Thus, If $\frac{dy}{dx}$ is the slope of tangent of the given family of curves, then the slope of the trajectories would be $\frac{-dx}{dy}$.

Working rule: (i) Diff. the eq^n and eliminate the constant term

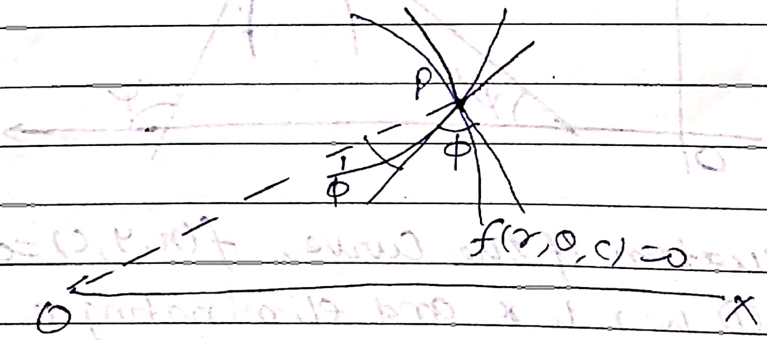
(ii) Replace $\frac{dy}{dx}$ by $\frac{-dx}{dy}$

(iii) Integrate the diff. eq, thus obtained and the the solution will be the required eq^n of orthogonal Trajectories.

(2) Polar Coordinates:-

To Find the orthogonal trajectories of the family of curves $f(r, \theta, c) = 0$, c being a parameter.

The procedure is exactly same as cartesian case.



Working rule:- (i) Diff. the equation and eliminate the constant term

(ii) Replace, $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

(iii) Integrate the new diff. eq^n thus obtained and the solution will be required eq^n of the orthogonal trajectories.